Valuation Rings

Recall that a submonoid Γ^+ of a group Γ defines an ordering on Γ, where $x \geq y$ iff $x - y \in \Gamma^+$. The set of elements x such that x and $-x$ both belong to Γ^+ is a subgroup; dividing by this subgroup we find that in the quotient $x \geq y$ and $y \geq x$ implies $x = y$. If this property holds already in Γ and if in addition, for every $x \in \Gamma$ either x or $-x$ belongs to Γ^+, the ordering defined by Γ^+ is total.

A *(non-Archimedean) valuation* of a field K is a homomorphism v from K^* onto a totally ordered abelian group Γ such that $v(x+y) \geq \min(v(x), v(y))$ for any pair x, y of elements whose sum is not zero. Associated to such a valuation is $R =: \{x : v(x) \in \Gamma^+\} \cup \{0\}$; one sees immediately that R is a subring of K with a unique maximal ideal, namely $\{x \in R : v(x) \neq 0\}$. This R is called the valuation ring associated with the valuation R.

Proposition 1 *Let R be an integral domain with fraction field K. Then the following are equivalent:*

1. *There is a valuation v of K for which R is the associated valuation ring.*
2. *For every element a of K, either a or a^{-1} belongs to R.*
3. *The set of principal ideals of R is totally ordered by inclusion.*
4. *The set of ideals of R is totally ordered by inclusion.*
5. *R is local and every finitely generated ideal of R is principal.*

Proof: (1) implies (2) is clear.

(2) implies (1) is easy: let Γ be the group K^*/R^* and let Γ^+ be the submonoid $R - \{0\}/R^*$.

(2) implies (3): This is clear, since the ordered set of principal ideals can be identified with the set ordered monoid $R - \{0\}/R^*$.

(3) implies (4): Suppose that I and J are ideals of R and I is not contained in J. Choose some $a \in I \setminus J$. Let b be any element of J. Since $a \notin J$, $a \not{i}n(b)$, and hence $(b) \subseteq (a) \subseteq I$ It follows that $J \subseteq I$.

(4) implies (5): Since the set of ideals of R is totally ordered, R has a unique maximal ideal, and hence is local. To prove that every finitely generated ideal is principal, it will suffice to show that any ideal which is

generated by two elements is in fact principle. But if $I = (f, g)$, either $(f) \subseteq (g)$ or $(g) \subseteq (f)$, so certainly $I = (f)$ or (g).

(5) implies (2). Suppose that a and b are elements of R, and let m be the maximal ideal of R, and let I be the (a, b). Then since I is principal, I/mI is a one-dimensional vector space over the field $k =: R/m$, and hence the images of a and b are linearly dependent. Thus we can find elements u and v of R such that $ua+vb \in mI$, with u and v not both in m. Furthermore, we can find x and y in m such that $ua + vb = xa + yb$, i.e. $a(u-x) = b(y-v)$. Now if for example u is a unit, so is $u-x$, and we see that $a/b = (y-v)/(u-x) \in R$. □

The following characterizations of valuations rings is more substantial.

Theorem 2 *Let R be an integral domain contained in a field K. Then the following conditions are equivalent:*

1. *R is a valuation ring of K.*
2. *R is local, and is maximal among local subrings of K under the partial ordering of domination.*
3. *There exists an algebraically closed field L and a homomorphism $\theta\colon R \to L$ (not necessarily injective) with respect to which R is maximal: if $R \subseteq R' \subseteq K$ and $\theta\colon R' \to L$ prolongs θ, then $R = R'$.*

Proof: (1) implies (2): Suppose that R is a valuation ring and is contained in a local subring R' of R. If $x \in R'$ and $x \notin R$, then $y =: x^{-1} \in R$. But then x^{-1} also belongs to R', so that y maps to a unit in R' and is not a unit in R; this means that R' does not dominate R.

(2) implies (3): Let k be the residue field of R, let $k \to L$ be an algebraic closure of k, and let $\theta\colon R \to L$ be the composite $R \to k \to L$. Suppose that R' is a subring of K containing R and $\theta'\colon R' \to L$ extends θ. Let m' be the kernel of θ'. Then θ' factors through the localization R'' of R' by θ', so we may as well replace R' by R'' and assume that R' is local, with maximal ideal m'. Since θ' prolongs θ, m maps to m', so R' dominates R. Then by assumption $R = R'$, as claimed.

(3) implies (1): First we prove a general lemma about ring extensions. Suppose that B is an A-algebra and b is an element of B. Recall that b is integral over A iff the subalgebra $A[b]$ of B generated by A and b is a finitely generated A-module. (Note that the map $A \to A[b]$ might not be injective.) We can also consider the subalgebra $A[b^{-1}]$ of B_b.

Lemma 3 *If b is any element of an A-algebra B, either b is integral over A, or $A[b^{-1}]/(b^{-1})$ is not zero.*

To see this, let $c =: b^{-1} \in B_b$, and suppose that the ring $A[c]/(c)$ is zero. Then c is a unit of $A[c]$, and hence one can find elements a_i of A such that $c(a_n c^n + a_{n-1}c^{n-1} + \cdots a_0) = 1$ in the ring B_b. Multiplying by b^{n+1}, we find that $a_n + a_{n-1}b + \cdots a_0 b^n = b^{n+1}$ in B_b, and it follows that there exists an integer $k \geq 1$ such that $a_n b^k + a_{n-1}b^{k+1} + \cdots a_0 b^{n+k-1} = b^{n+k}$ in B, proving that b is indeed integral over A. □

Now we can prove that (3) implies (1). First let us observe that (3) implies that R is a local ring, with maximal ideal the kernel of θ. Indeed, if $\theta(x)$ is not zero, then $\theta(x)$ is a unit of L, and hence θ prolongs to the localization of R by x. By the maximality property of θ, this localization is just R itself, so x is a unit of R. Now suppose that x is any element of K. If x is integral over R then $R' := R[x]$ is a finite extension of R and the maximal ideal m of R lifts to some maximal ideal m' of of R'. Then R'/m' is a finite extension of R/m and since L is algebraically closed, ϕ extends to R'/m' and hence to R'. Then $R' = R$ and $x \in R$. On the other hand, if x is not integral over R, the lemma implies that $y =: x^{-1}$ is not a unit of $R[y]$, so there is a maximal ideal m' of $R[y]$ containing y. Then $R \to R[y] \to R[y]/m'$ is surjective, and hence its kernel is a maximal ideal of R, which can only be the kernel of θ. This implies that θ prolongs to $R[y]$, and hence that $y \in R$. In paricular it follows that K is the fraction field of R and that R is a valuation ring of K. □

Corollary 4 *If R is a local ring contained in a field K, there exists a valuation ring R' of K which dominates R.*

Proof: Consider the family of local subrings of K with the partial ordering of domination. Thanks to the Hausdorff maximality principal, it will suffice to show that every chain has an upper bound. If $\mathcal{C}$ is such a chain, let V be the union of the elements of $\mathcal{C}$. It is clear that V is a subring of K. Furthermore, notice that if $a \in A \in \mathcal{C}$, then a is a unit of A iff it is a unit of V—for if $a^{-1} \in V$, then $a^{-1} \in B$ for some $B \in \mathcal{C}$, and since B dominates A, a must have already been a unit in A. Now it is clear that if x and y elements of V which aren't units, then $x + y$ is also not a unit, so that V is a local ring, and also that V dominates R. The corollary follows. □

Corollary 5 *Suppose that x and ξ are two points in a scheme X and that x is a specialization of ξ. Then there exists a valuation ring V and a morphism* $\operatorname{Spec} V \to X$ *which sends the generic point to ξ and the special point to x.*

Proof: Any affine neighborhood of x contains ξ, so we may as well assume that X is affine. Furthermore, x belongs to the closure of ξ, so we may replace X by this closure, i.e. we may assume that $X = \operatorname{Spec} A$, where A is an integral domain and ξ corresponds to the zero ideal. Let K be the fraction field of A and let $R \subseteq K$ be the localization of A at the ideal corresponding to x. Now by the previous corollary, there is a valuation ring V of K which dominates R. The map $A \to R \to V$ gives what we want. $\square$

Corollary 6 *If A is an integral domain contained in a field K, then the intersection of all the valuation rings V of K which contain A is precisely the integral closure $\tilde{A}$ of A in K.*

Proof: It is easy to see that a valuation ring is integrally closed in its field of fractions, and hence that $\tilde{A}$ is contained in every valuation ring containing A. If $x \in K$ is not integral over A, then let $y =: x^{-1}$; by the lemma we see that $A[y]/(y)$ is not zero. Let n be a maximal ideal of $A[y]$ containing y and let V be a valuation ring of K which dominates the localization of $A[y]$ at n. Then y belongs to the maximal ideal of V and hence x does not belong to V. $\square$

Now let's discuss discrete valuation rings.

Theorem 7 *Let V be an integral domain. The following are equivalent.*

1. *V is a noetherian valuation ring.*
2. *V is local and is a principal ideal domain.*
3. *V is noetherian and local and its maximal ideal is generated by a single element.*
4. *V is noetherian, local of Krull dimension less than or equal to one, and integrally closed in its field of fractions.*

Proof: In a valuation ring every finitely generated ideal is principal, so (1) implies (2), and a local ring is a valuation ring iff every finitely generated ideal is principal, so (2) implies (1). A principal ideal domain is noetherian, so (2) implies (3). To prove that (3) implies (2), suppose that π generates the maximal ideal of V. Since V is noetherian, $\cap m^i = 0$. If $a \neq 0$, there is then a k such that $a \in m^k \setminus m^{k+1}$, so $a = a'\pi^k$ with $a' \notin m$. Hence a' is a unit and $(a) = (\pi^k)$. Write $\nu(a)$ for this k. If I is a nonzero ideal of V, let ν be the minimum of $\nu(a)$ over all the nonzero elements $a \in I$. Then $I = (\pi^k)$, and hence is principal, concluding the proof of the equivalence of (1) through (3).

Since a principal ideal domain has dimension at most one and is integrally closed, (2) implies (4). It remains only to prove that (4) implies (3). If the maximal ideal m of V is zero, V is a field. Assume this is not the case. Since m is finitely generated, Nakayama's lemma implies that there is some element $\pi \in m \setminus m^2$. It will suffice to prove that $m = (\pi)$. Since V is integral, (0) is a prime ideal, and since V has dimension at most one, $(0) \subset m$ is a maximal chain of primes. Thus the dimension of V is in fact one, and since every prime ideal P contains (0) and is contained in m, V has exactly two prime ideals. Since $\pi \neq 0$, the quotient $V/(\pi)$ has only one prime ideal, namely $m/(\pi)$, and hence every element of $m/(\pi)$ is nilpotent. Since m is finitely generated, it follows that $m^i \subseteq (\pi)$ for some i. It will suffice to prove that the smallest such i is 1. Suppose that this i is at least 2. Choose $a \in m^{i-1}$. Then for any $x \in m$, $ax \in m^i$, and hence $ax = \pi y$ for some $y \in V$. Since $a \in m^{i-1}$ and $i \geq 2$, $a \in m$ and $ax \in m^2$. Since $\pi \notin m^2$, y is not a unit, so $y \in m$. We have just proved that multiplication by the element a/π of the fraction field K of V maps m into itself. But m is a finitely generated faithful V-module, and it follows from this that a/π is integral over V. Since V is integrally closed in K, it follows that $a/\pi \in V$, *i.e.*, $a \in (\pi)$. Thus we have proved that $m^{i-1} \subseteq (\pi)$, contradicting the minimality of i.

$\square$